Rank-1 Games With Exponentially Many Nash Equilibria

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Abstract

The rank of a bimatrix game (A,B) is the rank of the matrix $A+B$. We give a construction of rank-1 games with exponentially many equilibria, which answers an open problem by Kannan and Theobald (2010).

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Finding a Nash equilibrium of a bimatrix game is a PPAD-complete problem (Chen and Deng, 2006). For that reason, classes of bimatrix games where a Nash equilibrium can be found more easily are of some interest. An equilibrium of a zero-sum game (A,B) where $A+B$ is the all-zero matrix can be found in polynomial time by solving a linear program. As a generalization, Kannan and Theobald (2010) defined the *rank* of a bimatrix game (A,B) as the rank of the matrix $A+B$, and give a polynomial-time algorithm to find an approximate equilibrium of a game of fixed rank. They asked (Open Problem 9) if a rank-1 game may possibly have only a polynomial number of Nash equilibria. This is not the case, according to the following theorem.

Theorem 1 *Let $p>2$ and let (A,B) be the $n\times n$ bimatrix game with entries of A*

$$a_{ij} = \begin{cases} 2p^{i+j} & \text{if } j>i \\ p^{2i} & \text{if } j=i \\ 0 & \text{if } j<i \end{cases} \tag{1}$$

for $1\le i,j\le n$, and $B=A^\top$. Then $A+B$ is of rank 1, and (A,B) is a nondegenerate bimatrix game with 2^n-1 many Nash equilibria.

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Proof. By (1), $A+B=\alpha\beta^\top$ with the n components of the column vectors α and β defined by $\alpha_i = p^i$ and $\beta_j = 2p^j$ for $1 \le i, j \le n$, so $A+B$ is of rank 1.

Let y be any mixed strategy of the column player and let S be the support of y, that is, $S = \{j \mid y_j > 0\}$. Consider any row i and let $T = \{j \in S \mid j > i\}$. Then, because A is upper triangular, the expected payoff against y in row i is

$$(Ay)_i = a_{ii}y_i + \sum_{j\in T} a_{ij}y_j\,. \tag{2}$$

Suppose $i \notin S$. If T is empty, then $(Ay)_i = 0 < (Ay)_1$, otherwise let $t = \min T$ and note that for $j \in T$ we have $a_{ij} = 2p^{i+j} < p^{1+i+j} \le p^{t+j} \le a_{tj}$, so $(Ay)_i < (Ay)_t$. Hence, no row i outside S is a best response to y. Similary, because the game is symmetric, any column that is a best response to a strategy x of the row player belongs to the support of x. So no mixed strategy has more pure best responses than the size of its support, that is, the game is nondegenerate (von Stengel, 2002). Moreover, if (x,y) is a Nash equilibrium of (A,B), then x and y have equal supports.

For any nonempty subset S of $\{1,\ldots,n\}$, we construct a mixed strategy y with support S so that (y,y) is a Nash equilibrium of (A,B). This implies that the game has 2^n-1 many Nash equilibria, one for each support set S. The equilibrium condition holds if $(Ay)_i = u$ for $i \in S$ with equilibrium payoff u, because then $(Ay)_i < u$ for $i \notin S$ as shown above. We start with $s = \max S$, where $(Ay)_s = a_{ss}y_s = u$, by fixing u as some positive constant (e.g., $u=1$), which determines y_s. Once y_i is known for all $i \in S$ (and $y_i = 0$ for $i \notin S$), we scale y and u by multiplication with $1/\sum_{i\in S} y_i$ so that y becomes a mixed strategy. Assume that $i \in S$ and $T = \{j \in S \mid j > i\} \ne \emptyset$ and assume that y_k has been found for all k in T so that $(Ay)_k = u$ for all k in T, which is true for $T = \{s\}$. Then, as shown above, $\sum_{j\in T} a_{ij}y_j < \sum_{j\in T} a_{tj}y_j = (Ay)_t = u$ for $t = \min T$, so y_i is determined by $(Ay)_i = u$ in (2), and $y_i > 0$. By induction, this determines y_i for all i in S, and after re-scaling gives the desired equilibrium strategy y. $\square$

Adsul, Garg, Mehta, and Sohoni (2011) showed how to find in polynomial time an exact Nash equilibrium of a rank-1 game, which is of the form $(A, -A+\alpha\beta^\top)$ with suitable column vectors $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}^n$. They proved that a mixed strategy pair (x,y) is a Nash equilibrium of this game if and only if for some suitable real λ the equation $x^\top\alpha = \lambda$ holds and (x,y) is a Nash equilibrium of the game $(A, -A+\mathbf{1}\lambda\beta^\top)$, where $\mathbf{1}$ is the all-one vector; this equilibrium can be found as the solution to a linear program parameterized by λ. Their algorithm uses binary search for λ combined with solving the parameterized LP.

The exponential number of Nash equilibria of the game in Theorem 1 shows that the path that follows the solutions of the parameterized LP with parameter λ has an exponential number of intersections with the hyperplane defined by $x^\top\alpha = \lambda$. Hence, that path has exponentially many line segments. Murty (1980) describes a parameterized LP with such an exponentially long path of length 2^n. His LP is of the form

$$\text{maximize } c^\top z \quad \text{subject to } Az \ge b + \mathbf{1}\lambda,\ \ z \ge 0 \tag{3}$$

with A as in (1) with $p=1$, and the vectors c and b in $\mathbb{R}^n$ given by $c_j = 4^{n-j}$ and $b_i = -2^{n-i}$ for $1 \le i, j \le n$. The payoffs for the game in Theorem 1 have been inspired

by Murty's example, but are not systematically constructed from it; at the point of this writing, it is not even clear how to get a game with that many equilibria from Murty's result.

For specific instances of the game in Theorem 1 one can choose $p = 3$ or $p = 4$ in (1) and divide all payoffs by p^2 (or let the rows and columns be numbered from 0 to $n-1$ rather than 1 to n). In the construction of mixed strategies y with support S described in the proof, starting with $u = p^s$ then gives integer values for y_i for $i \in S$ which are afterwards re-scaled. Verifying the equilibria of these games was aided by the webpage of Savani (2012).

The number of $2^n - 1$ Nash equilibria of an $n \times n$ bimatrix game is large, the same as that of the coordination game where both players have the identity matrix (which has maximal rank). Quint and Shubik (1997) even conjectured this to be the maximum possible number (always considering nondegenerate games), which is true for $n \leq 4$ (Keiding, 1997; McLennan and Park, 1999). However, this conjecture was refuted by von Stengel (1999) who constructed a 6×6 game with 75 equilibria, and more generally $n \times n$ games with asymptotically more than 2.4^n equilibria. Quint and Shubik (2002) showed that a game (A,A) where both players have identical payoffs has at most $2^n - 1$ equilibria. A *symmetric* game $(A,A^\top)$ of size $n \times n$, as considered in Theorem 1, has at most $2^n - 1$ symmetric equilibria, because an equilibrium is uniquely determined by the pair of supports for the two strategies. However, the number of possibly nonsymmetric equilibria of a symmetric game is not bounded by $2^n - 1$, as the following simple argument based on a standard symmetrization shows. Suppose (A,B) is an $n \times n$ bimatrix game with positive payoff matrices and more than 2^n equilibria, and let $C = \begin{pmatrix} 0 & A \\ B^\top & 0 \end{pmatrix}$. Then for any *pair* of equilibria (x,y), (x',y') of (A,B), one obtains an equilibrium $((\hat{x},\hat{y}'),(\hat{x}',\hat{y}))$ of $(C,C^\top)$ where $\hat{x}$, $\hat{x}'$, $\hat{y}$, and $\hat{y}'$ are scaled versions of x, x', y, and y', respectively, so that the respective optimal payoffs of $A\hat{y}$ and $B^\top\hat{x}'$ coincide, and similarly those of $B^\top\hat{x}$ and $A\hat{y}'$. Then $(C,C^\top)$ is of size $2n \times 2n$ and has more than $(2^n)^2$ many equilibria, as claimed.

Hence, it is an open question if there are nondegenerate $n \times n$ games of rank 1 with more than 2^n many Nash equilibria.

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